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Applications of a Z_p Index Theory to Periodic Solutions for a Class of Functional Differential Equations[1]

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By means of a new geometrical index with Z_p group actions, multiplicity results for a certain class of nonautonomous time periodic functional differential systems are obtained. © 2001 Academic Press

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In this paper, we introduce a new geometrical Z_p index theory and apply it to investigate the second-order functional differential equation

$$\frac{d^2x}{dt^2} + f(t, x(t-r_1), \ldots, x(t-r_s)) = 0,$$

where $x = (x_1, x_2, \ldots, x_N)^T \in \mathbf{R}^N$, $f(t+T, x^{(1)}, \ldots, x^{(s)}) = f(t, x^{(1)}, \ldots, x^{(s)})$, $\forall x^{(j)} \in \mathbf{R}^N$, and $r_j = \tau_j T$, τ_j integers, $j = 1, 2, \ldots, s$.

The purpose of this paper is to establish a multiplicity result of periodic solutions for above nonautonomous functional differential systems. Our approach takes advantage of symmetry techniques and variational structures, which allows us to obtain better estimates on multiple critical points of the dual functionals. Especially, the multiplicity result is new even for ordinary differential systems. In the case of second order ordinary differential systems, our result has improved the conclusions of [5, 6] and gives a better estimate for the number of subharmonic solutions.

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1. A NEW GEOMETRICAL INDEX THEORY

In this section, we fix p as a positive integer, and $p > 1$. Let X be a Banach space and let μ be a linear isometric action of Z_p on X. Namely, $\mu \in \mathscr{L}(X, X)$, $\|\mu x\| = \|x\|$, $\forall x \in X$, and $\mu^p = \mathrm{id}$, where Z_p is a cyclic group with order p. A subset of A of X will be called μ-invariant if $\mu(A) \subset A$. A continuous map f: $A \to X$ is called μ-equivariant if $f(\mu x) = \mu f(x)$, $\forall x \in A$.

Set

$$\Sigma = \{A \subset X \text{ is closed and } \mu\text{-invariant}\}.$$

For any given positive integer m, we denote by E_m the set of all divisors of m.

For any set $A \in \Sigma$ and positive integer k, we call φ: $A \to C^k$ a $(\mu, E_m)^k$-type map, if φ is continuous and there exist $m_1, \ldots, m_k \in E_m$, such that $\varphi_j(\mu x) = e^{i(2\pi m_j/p)}\varphi_j(x), \forall x \in A$, $j = 1, 2, \ldots, k$, where $\varphi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_k(x))$.

Set

$$Y = \{z \in \mathbf{C} \mid \arg z = \frac{2\pi j}{p}, j = 0, 1, \ldots, p-1\}.$$

Now, we define our index i_m: $\Sigma \to \mathbf{N} \cup \{+\infty\}$, $\forall A \in \Sigma$ as follows:

$$i_m(A) = \min\big\{k \in \mathbf{N} \mid \text{there exists a } (\mu, E_m)^k\text{-type map } \varphi\colon A \to Y^k \backslash \{\theta\}\big\}. \tag{1.1}$$

Set $i_m(A) = 0$ if $A = \varnothing$ and set $i_m(A) = +\infty$ if no such map exists.

First, we will show in the following theorem that i_m defined as above is really an index according to the definition of [2].

THEOREM 1.1. *The index i_m, as usual, has following properties*:

(i) $i_m(A) = 0 \Longleftrightarrow A = \varnothing$.

(ii) (*monotonicity*) *If ψ: $A \to B$ is a continuous μ-equivariant map, then $i_m(A) \le i_m(B)$, where $A, B \in \Sigma$.*

(iii) (*subadditivity*) $i_m(A \cup B) \le i_m(A) + i_m(B)$.

(iv) (*superinvariance*) $i_m(A) \le i_m(\overline{h(A)})$, *where h is continuous and μ-equivariant*.

(v) (*continuity*) $\forall A \in \Sigma$, *if A is a compact set, then there exists a closed neighborhood $\Omega(A)$ of A such that $i_m(\Omega(A)) = i_m(A)$.*

We only need to prove property (iii) and the others are simple. The following lemma seems to be necessary.

LEMMA 1.1. *Let $A \in \Sigma$, φ: $A \to Y^k \backslash \{\theta\}$ be a $(\mu, E_m)^k$-type map; then φ can be extended to $\tilde{\varphi}$: $X \to Y^k$, where $\tilde{\varphi}$ is also a $(\mu, E_m)^k$-type map and $\tilde{\varphi}|_A = \varphi$.*

Proof. By hypothesis, there exist $m_1, \dots, m_k \in E_m$, such that

$$\varphi_j(\mu x) = e^{i(2\pi m_j/p)} \varphi_j(x)$$

$\forall x \in A$ and $j = 1, 2, \dots, k$.

By Tietze's extension theorem and averaging on Z_p, we have an extension of φ, written $\bar{\varphi}$: $X \to \mathbf{C}^k$, satisfying $\bar{\varphi}_j(\mu x) = e^{i(2\pi m_j/p)} \bar{\varphi}_j(x)$, $j = 1, 2, \dots, k$, $x \in X$, and $\bar{\varphi}|_A = \varphi$.

To complete the proof of Lemma 1.1, we use a kind of special decomposition of complex numbers (cf. [5], etc.), namely, $\forall z \in \mathbf{C}$, there exist unique $U(z)$ and $V(z)$ such that $z = U(z) + e^{i\pi/p} V(z)$, where U, V: $\mathbf{C} \to Y$ are continuous and satisfy

(a) $z = 0 \Longleftrightarrow U(z) = V(z) = 0$;

(b) $z \in Y \Longleftrightarrow U(z) = z$ and $V(z) = 0$;

(c) $U(e^{i(2\pi/p)} z = e^{i(2\pi/p)} U(z)$, $V(e^{i(2\pi/p)} z) = e^{i(2\pi/p)} V(z)$.

Now, let $\tilde{\varphi}(x) = (U(\bar{\varphi}_1(x)), \dots, U(\bar{\varphi}_k(x)))$; then $\tilde{\varphi}$: $X \to Y^k$ is continuous and it is easy to see that $\tilde{\varphi}_j(\mu x) = e^{i(2\pi m_j/p)} \tilde{\varphi}_j(x)$, $j = 1, 2, \dots, k$, $\forall x \in X$, and $\tilde{\varphi}|_A = \varphi$.

Proof of property (iii). We assume $A, B \in \Sigma$, $i_m(A) = k$, $i_m(B) = l$.

By definition, there exist two continuous maps φ, ψ and $m_1, \dots, m_k \in E_m$, $n_1, \dots, n_l \in E_m$, such that

$$\varphi\colon A \to Y^k \backslash \{\theta\}, \qquad \psi\colon B \to Y^l \backslash \{\theta\}$$

and

$$\varphi_j(\mu x) = e^{i(2\pi m_j/p)} \varphi_j(x), \qquad \psi_s(\mu x) = e^{i(2\pi n_s/p)} \psi_s(x),$$

$j = 1, 2, \dots, k$, $s = 1, 2, \dots, l$.

According to Lemma 1.1., we have continuous extensions of φ and ψ, denoted by $\tilde{\varphi}, \tilde{\psi}$, respectively, where $\tilde{\varphi}$: $X \to Y^k$, $\tilde{\psi}$: $X \to Y^l$, and they are $(\mu, E_m)^k$-type and $(\mu, E_m)^l$-type maps, respectively.

Set

$$\eta(x) = (\tilde{\varphi}(x), \tilde{\psi}(x)).$$

We know $\eta|_{A\cup B}$: $A \cup B \to Y^{k+l} \backslash \{\theta\}$. Furthermore, it is a $(\mu, E_m)^{k+l}$-type map. By the formula (1.1) we have $i_m(A \cup B) \le i_m(A) + i_m(B)$. Thus, the proof is completed.

$\forall x \in X$, we denote by $[x]$ the Z_p-orbit of x, i.e.,

$$[x] = \{\mu^j x \mid j = 0, 1, \dots, p-1\}.$$

The index i_m also has normallity under certain conditions. This is the following theorem.

THEOREM 1.2. *$\forall x \in X$, let $k = \min\{j \in \mathbf{N} : \mu^j x = x\}$. If $k \neq 1$ and $p/k \in E_m$, then $i_m([x]) = 1$.*

Proof. Let $l = p/k$; then $l \in E_m$. We define a map φ: $[x] \to Y\backslash\{\theta\}$ as follows:

$$\varphi(\mu^j x) = e^{i(2\pi lj/p)}, \qquad j = 1, 2, \ldots, k.$$

Then

$$\varphi(\mu\mu^j x) = e^{i(2\pi l(j+1/p)} = e^{i(2\pi l/p)}\varphi(\mu^j x), \qquad j = 1, 2, \ldots, k-1,$$
$$\varphi(\mu\mu^k x) = \varphi(\mu x) = e^{i(2\pi l/p)} = e^{i(2\pi l/p)}\varphi(\mu^k x).$$

So, $i_m([x]) \leq 1$. Obviously, $i_m([x]) \geq 1$. Thus $i_m([x]) = 1$.

The most important aspect of an index theory lies in the applications of its dimensional property to estimate the critical point numbers of certain kinds of functionals. The following discussions show that the index i_m possesses a dimensional property under some appropriate conditions.

Denote by X_{2a} the μ-invariant linear subspace of X with dimension $2a$. We identify X_{2a} with $\mathbf{C}^a$ by $x = (x_1, x_2, \ldots, x_{2a}) \in X_{2a}$ with

$$(x_1 + ix_{a+1}, \ldots, x_a + ix_{2a}) \in \mathbf{C}^a.$$

A Z_p action μ on X_{2a} is given by

$$\mu z = (e^{i(2\pi k_1/p)} z_1, \ldots, e^{i(2\pi k_a/p)} z_a)$$

for $z = (z_1, \ldots, z_a) \in \mathbf{C}^a$, where $k_j \neq 0$ integers, $j = 1, 2, \ldots, a$.

We denote by $\langle n_1, n_2, \ldots, n_s\rangle$ and $[n_1, n_2, \ldots, n_s]$ the greatest common divisor and the smallest common multiple of s positive integers $n_1, \ldots, n_s$, respectively.

PROPOSITION 1.1. *Assume that an isometric action μ on $\mathbf{C}^a$ with group Z_p is given by*

$$\mu z = (e^{i(2\pi k_1/p)} z_1, e^{i(2\pi k_2/p)} z_2, \ldots, (e^{i(2\pi k_a/p)} z_a)$$

for $z = (z_1, z_2, \ldots, z_a) \in \mathbf{C}^a$, where $k_j \neq 0$ integers, $j = 1, 2, \ldots, a$. Let Ω be a closed μ-invariant neighborhood of θ in $\mathbf{C}^a$. Assume further that F: $\partial\Omega \to \mathbf{C}^a\backslash\{\theta\}$ is a continuous map and there exist nonzero integers $m_1, \ldots, m_a$, such that $\forall z \in \partial\Omega$, $F_j(\mu z) = (e^{i(2\pi m_j/p)} F_j(z)$, $j = 1, 2, \ldots, a$. Then

$$\deg(F, \Omega, \theta) = \prod_{j=1}^{a} \left(\frac{m_j}{k_j}\right)\left[1 + \alpha\left(\frac{p}{\langle p, K, M\rangle}\right)\left(\frac{\langle K, M\rangle}{M}\right)^a\right], \tag{1.2}$$

where $K = \langle |k_1|, |k_2|, \ldots, |k_a|\rangle$, $M = [|m_1|, |m_2|, \ldots, |m_a|]$, and α is an integer depending on F.

The proof of this proposition can be referred to [10].

THEOREM 1.3 (the dimensional property of index i_m). *Let Ω be a open bounded μ-invariant neighborhood of θ in X_{2a}. If $m^a \neq 0 (\mathrm{mod}\, p)$, then*

$$i_m(\partial\Omega) = 2a,$$

where

$$m = [|k_1|, |k_2|, \dots, |k_a|].$$

Proof. We regard Ω as a open bounded μ-invariant domain in $\mathbf{C}^a$ which contains θ. Define $\varphi: \partial\Omega \to Y^{2a} \backslash \{\theta\}$ by

$$\varphi(z) = \big(U(z_1^{q_1}), V(z_1^{q_1}), \dots, U(z_a^{q_a}), V(z_a^{q_a})\big),$$

$\forall z = (z_1, z_2, \dots, z_a) \in \partial\Omega$, where q_j is some integer satisfying $\langle |k_j|, p\rangle = k_j q_j (\mathrm{mod}\, p)$, $j = 1, 2, \dots, a$. The expressions of U and V can be found in the proof of Lemma 1.1.

Set $l_j = \langle |k_j|, p\rangle$, $j = 1, 2, \dots, a$; then $l_j \in E_m$, where

$$m = [|k_1|, |k_2|, \dots, |k_a|].$$

Thus,

$$\begin{aligned}
\varphi(\mu z) &= \varphi\big(e^{i(2\pi k_1/p)} z_1, e^{i(2\pi k_2/p)} z_2, \dots, e^{i(2\pi k_a/p)} z_a\big) \\
&= \big(U\big(e^{i(2\pi k_1 q_1/p)} z_1^{q_1}\big), V\big(e^{i(2\pi k_1 q_1/p)} z_1^{q_1}\big), \dots, \\
&\qquad U\big(e^{i(2\pi k_a q_a/p)} z_a^{q_a}\big), V\big(e^{i(2\pi k_a q_a/p)} z_a^{q_a}\big)\big) \\
&= \big(U\big(e^{i(2\pi l_1/p)} z_1^{q_1}\big), V\big(e^{i(2\pi l_1/p)} z_1^{q_1}\big), \dots, \\
&\qquad U\big(e^{i(2\pi l_a/p)} z_a^{q_a}\big), V\big(e^{i(2\pi l_a/p)} z_a^{q_a}\big)\big) \\
&= \big(e^{i(2\pi l_1/p)} U\big(z_1^{q_1}\big), e^{i(2\pi l_1/p)} V\big(z_1^{q_1}\big), \dots, \\
&\qquad e^{i(2\pi l_a/p)} U\big(z_a^{q_a}\big), e^{i(2\pi l_a/p)} V\big(z_a^{q_a}\big)\big) \\
&= \big(e^{i(2\pi l_1/p)} \varphi_1(z), e^{i(2\pi l_1/p)} \varphi_2(z), \dots, \\
&\qquad e^{i(2\pi l_a/p)} \varphi_{2a-1}(z), e^{i(2\pi l_a/p)} \varphi_{2a}(z)\big).
\end{aligned}$$

Because of $l_j \in E_m$, $j = 1, 2, \dots, a$, by the definition of index i_m, we have $i_m(\partial\Omega) \leq 2a$.

On the other hand, if $i_m(\partial\Omega) < 2a$, then there must exist $m_j \in E_m$, $j = 1, 2, \dots, 2a-1$ and a continuous map $\varphi: \partial\Omega \to Y^{2a-1} \backslash \{\theta\}$ satisfying

$$\varphi_j(\mu z) = e^{i(2\pi m_j/p)} \varphi_j(z), \qquad j = 1, 2, \dots, 2a-1.$$

By Lemma 1.1, we have a extension of φ denoted by $\tilde{\varphi}$: $\mathbf{C}^a \to Y^{2a-1}$, which satisfies $\tilde{\varphi}|_{\partial\Omega} = \varphi$ and

$$\tilde{\varphi}_j(\mu z) = e^{i(2\pi m_j/p)}\tilde{\varphi}_j(z), \qquad j = 1, 2, \ldots, 2a-1.$$

We denote by l_j the smallest common multiple of m_{2j-1} and m_{2j}; i.e., $l_j = [m_{2j-1}, m_{2j}]$, $j = 1, 2, \ldots, a-1$, $l_a = m_{2a-1}$. Then $l_j \in E_m$, $j = 1, 2, \ldots, a$.

Define map F: $\mathbf{C}^a \to \mathbf{C}^a$ as follows:

$$\begin{aligned}F(z) = \big((\tilde{\varphi}_1(z))^{l_1/m_1} + e^{i(\pi/2p)}(\tilde{\varphi}_2(z))^{l_1/m_2}, \ldots, (\tilde{\varphi}_{2a-3}(z))^{l_{a-1}/m_{2a-3}} \\ + e^{i(\pi/2p)}(\tilde{\varphi}_{2a-2}(z))^{l_{a-1}/m_{2a-2}}, \tilde{\varphi}_{2a-1}(z)\big).\end{aligned}$$

Then

$$\begin{aligned}F_j(\mu z) &= (\tilde{\varphi}_{2j-1}(\mu z))^{l_j/m_{2j-1}} + e^{i(\pi/2p)}(\tilde{\varphi}_{2j}(\mu z))^{l_j/m_{2j}} \\ &= e^{i(2\pi l_j/p)}[\tilde{\varphi}_{2j-1}(z)^{l_j/m_{2j-1}} + e^{i(2\pi l_j/p)}e^{i(\pi/2p)}[\tilde{\varphi}_{2j}(z)]^{l_j/m_{2j}} \\ &= e^{i(2\pi l_j/p)}F_j(z),\end{aligned}$$

$j = 1, 2, \ldots, a-1$, and

$$F_a(\mu z) = \tilde{\varphi}_{2a-1}(\mu z) = e^{i(2\pi m_{2a-1}/p)}\tilde{\varphi}_{2a-1}(z) = e^{i(2\pi l_a/p)}F_a(z).$$

Thus,

$$F_j(\mu z) = e^{i(2\pi l_j/p)}F_j(z), \qquad j = 1, 2, \ldots, a.$$

Clearly $\theta \notin F(\partial\Omega)$. By Proposition 1.1, we have

$$\deg(F, \Omega, \theta) = \prod_{j=1}^{a}\left(\frac{l_j}{k_j}\right)\left[1 + \alpha\left(\frac{p}{\langle p, K, L, \rangle}\right)\left(\frac{\langle K, L\rangle}{L}\right)^a\right],$$

where $L = [|l_1|, \ldots, |l_\alpha|]$ and the other expressions are the same as those appearing in Lemma 1.2. Since $L = [|l_1|, \ldots, |l_\alpha|]$, $l_j \in E_m$, $j = 1, 2, \ldots, a$, we have $L \in E_m$. Moreover, by the hypotheses of Theorem 1.3, $m^a \neq 0 (\mathrm{mod}\ p)$, so $L^a \neq 0 (\mathrm{mod}\ p)$. Thus, we have $\deg(F, \Omega, \theta) \neq 0$. On the other hand, $F_a(z) = \tilde{\varphi}_{2a-1}(z) \in Y$. By the continuity of Brouwer degree, we can find $\delta \in \mathbf{C}\backslash Y$, $|\delta|$ sufficiently small, such that

$$\deg(F, \Omega, \bar{z}) \neq 0,$$

where $\bar{z} = (0, \ldots, 0, \delta)$. However, for $\forall(z) \in \overline{\Omega}$, $F(z) \neq \bar{z}$. Hence, $\deg(F, \Omega, \bar{z}) = 0$. This leads to a contradiction we finish our proof.

Remark. The above defined index i_m is a generalization of the Z_2 index (cf. [2]) and the Z_p index (cf. [5, 12]). This is clear for the Z_2 index. As for the Z_p index σ_n defined in [5, 12], we only need to set $m_1 = \cdots = m_k = m$. The Z_p index introduced in [6, 11] differs from i_m. However, due to its better dimensional property and more convenient applications, i_m has more advantages than other indices, which will be shown in the following discussions.

We now give an example of calculating the index of a certain invariant set which will be used later.

Let

$$X = \Big\{ u \in L^\alpha([0, 2\pi p], \mathbf{R}^N) : u(t) \text{ is } 2\pi p - \text{periodic function} \\ \text{and } \int_0^{2\pi p} u(s)\,ds = 0 \Big\}, \quad (1.3)$$

where $\alpha > 1$. A linear isometric action μ of Z_p on Banach space X is defined by

$$\mu(\mu(t)) = u(t + 2\pi).$$

In the following, we denote by $\operatorname{scm}(l)$ the smallest common multiple of integers $1, 2, \ldots, l$, and for a given $p > 1$,

$$n \triangleq \max\{l \in \mathbf{N} \mid [\operatorname{scm}(l)]^{lN} \neq 0 (\operatorname{mod} p)\} \quad (1.4)$$

$$m \triangleq \operatorname{scm}(n). \quad (1.5)$$

In the sequel, without any specification, n, m will always be defined as above.

For a given closed μ-invariant subset of X, we define $i_m(A)$ as (1.1). Consider the $2nN$-dimensional subspace X_{2nN} of X

$$X_{2nN} = Y_1 \otimes Y_2 \otimes \cdots \otimes Y_n,$$

where

$$Y_j = \operatorname{span}\Big\{ \sin\frac{jt}{p} e_k, \cos\frac{jt}{p} e_k, k = 1, 2, \ldots, N \Big\}$$

and $e_1 = (1, 0, \ldots, 0), \ldots, e_N = (0, \ldots, 0, 1)$.

For any $u(t) \in X_{2nN}$, we may express $u(t)$ as

$$u(t) = \sum_{|j|=1}^{n} \xi_j e^{i(jt/p)}, \qquad \xi_j \in \mathbf{C}^N, \qquad \xi_{-j} = \bar{\xi}_j. \quad (1.6)$$

Then $\forall u(t) \in X_{2nN}$,

$$\mu(u(t)) = \sum_{|j|=1}^{n} \xi_j e^{i(2\pi j/p)} e^{i(jt/p)}.$$

Under the identification $u(t) = \sum_{|j|=1}^{n} \xi_j e^{i(jt/p)}$, $\xi_j \in \mathbf{C}^N$, $\xi_{-j} = \bar{\xi}_j$ in X_{2nN} with $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ in $\mathbf{C}^{nN}$, we get the representation of μ on $\mathbf{C}^{nN}$ as

$$\mu(\xi) = (e^{i(2\pi/p)}\xi_1, \ldots, e^{i(2\pi n/p)}\xi_n),$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbf{C}^{nN}$, $\xi_j \in \mathbf{C}^N$, $j = 1, 2, \ldots, n$.

$\forall \varepsilon > 0$, $\eta > 0$, we take a subset of X_{2nN} as follows:

$$S_{n,\eta,\varepsilon} = \left\{ u(t) \in X_{2nN} : u(t) = \sum_{|j|=1}^{n} \xi_j e^{i(jt/p)}, \xi_{-j} = \bar{\xi}_j, \right.$$
$$\left. |\xi_1| = \eta, \sum_{j=2}^{n} |\xi_j| = \varepsilon \right\}. \tag{1.7}$$

In view of the conclusion of Theorem 1.3 and the meaning of m, n, we get

THEOREM 1.4. $i_m(S_{n,\eta,\varepsilon}) = 2nN$.

2. THE APPLICATIONS OF INDEX i_m TO FUNCTIONAL DIFFERENTIAL EQUATIONS

Since we have proved the dimensional property of index i_m (i.e., Theorem 1.3) in the last section, we are in a position to exploit it to investigate the existence of subharmonic solutions of second order time periodic functional differential equations.

Consider the second order functional differential equation

$$\frac{d^2x}{dt^2} + f(t, x(t - r_1), \ldots, x(t - r_s)) = 0, \tag{2.1}$$

where $x = (x_1, x_2, \ldots, x_N)^T \in \mathbf{R}^N$, $f(t + T, x^{(1)}, \ldots, x^{(s)}) = f(t, x^{(1)}, \ldots, x^{(s)})$, $\forall x^{(j)} \in \mathbf{R}^N$ and $r_j = \tau_j T$, τ_j integer, $j = 1, 2, \ldots, s$.

Assume $\tau = \langle |\tau_1|, \ldots, |\tau_s| \rangle$, $p \mid \tau$ and denote $q = \tau/p$.

Now we make the following assumptions on f.

(H_1) $f(t, x, \ldots, x)$ is continuous in $(t, x) \in \mathbf{R} \times \mathbf{R}^N$ and

$$\int_{-\infty}^{x} f(t, s, \ldots, s)ds < +\infty, \qquad \forall (t, x) \in \mathbf{R} \times \mathbf{R}^N.$$

(H_2) For $\forall t \in [0, T]$, $\int_{-\infty}^{x} f(t, s, \ldots, s)\, ds$ is convex and there exist constants $\alpha_1 > 0, \alpha_2 > 0$ and $1 < \beta < 2$, such that

$$\frac{\alpha_1}{\beta}|x|^{\beta} \leq \int_{-\infty}^{x} f(t, s, \ldots, s)\, ds \leq \frac{\alpha_2}{\beta}|x|^{\beta},$$

$\forall x \in \mathbf{R}^N$.

(H_3) If $x = x(t)$ is a periodic function with minimal positive period qT, q is a rational number, and $f(t, x(t), \ldots, x(t))$ is also a periodic function with minimal positive period qT, then q must be an integer.

For simplicity, we denote $F(t, x) = \int_{-\infty}^{x} f(t, s, \ldots, s)\, ds$ and S_p the smallest prime divisor of p.

Our main result is the following theorem.

THEOREM 2.1. *Under the hypotheses* (H_1), (H_2), *and* (H_3), *for any given integer* $p > 1$, *if* $p \mid \tau$ *and*

$$\frac{\alpha_2}{\alpha_1} < \left(\frac{S_p}{\sqrt{2}}\right)^{\beta},$$

then there exist at least $2nN$ *different periodic solutions of system* (2.1) *with minimal positive period* pT, *where*

$$n = \max\{l \in \mathbf{N} \mid [\mathrm{scm}(l)]^{lN} \not\equiv 0 (\mathrm{mod}\ p)\},$$
$$\tau = \langle |\tau_1|, \ldots, |\tau_s| \rangle.$$

The proof of Theorem 2.1 will be given by using a variational formulation, a minimax method, and the index theory introduced in Section 1. For simplicity, we shall restrict our attention to the case of F strictly convex in x. By an argument similar to that in [6], it is easy to verify that our results are also true for convex F. Without loss of generality, we set $T = 2\pi$.

Denote by F^* the partial Fenchel conjugate of F (with respect to the variable x); i.e.,

$$F^*(t, u) = \sup_{x \in \mathbf{R}^N} \{(u, x) - F(t, x)\}, \tag{2.2}$$

where $(\cdot, \cdot)$ is the usual scalar product in $\mathbf{R}^N$. By standard convex analysis we have

(i)* $F^*(t, \cdot)$ is convex and $F^*(\cdot, u)$ is 2π-periodic;

(ii)* $\alpha = \frac{\beta}{\beta - 1} > 2$ and

$$\frac{1}{\alpha}\left(\frac{1}{\alpha_2}\right)^{\alpha-1}|u|^{\alpha} \leq F^*(t, u) \leq \frac{1}{\alpha}\left(\frac{1}{\alpha_1}\right)^{\alpha-1}|u|^{\alpha},$$

$\forall t \in \mathbf{R}$, $\forall u \in \mathbf{R}^N$.

Set

$$X=\left\{u\in L^{\alpha}([0,2\pi p],\mathbf{R}^N)\;\middle|\;\int_0^{2\pi p}u(s)\,ds=0\right\},$$

with $\alpha=\frac{\beta}{\beta-1}$. Define the operator $\mathscr{K}$: $X\to X$ as

$$\begin{aligned}\mathscr{K}u(t)=&\int_0^t ds\int_0^s u(r)dr-\frac{t}{2\pi p}\int_0^{2\pi p}ds\int_0^s u(r)dr-\frac{1}{2\pi p}\int_0^{2\pi p}\\&\times dt\left[\int_0^t ds\int_0^s u(r)dr-\frac{t}{2\pi p}\int_0^{2\pi p}ds\int_0^s u(r)dr\right],\qquad\forall u\in X.\end{aligned}\tag{2.3}$$

It is easy to verify that $\mathscr{K}$ is a compact self-adjoint operator in X and satisfies

$$\mathscr{K}u(0)=\mathscr{K}u(2\pi p),\qquad\frac{d}{dt}\mathscr{K}u(0)=\frac{d}{dt}\mathscr{K}u(2\pi p),$$

and

$$\frac{d^2}{dt^2}\mathscr{K}u=u.$$

The spectrum $\sigma(\mathscr{K})$ is given by

$$\sigma(\mathscr{K})=\{-p^2/j^2\mid j\in\mathbf{N}\}\cup\{0\}.$$

The eigenspace corresponding to the eigenvalue $-p^2/j^2$ is

$$X_j=\operatorname{span}\left\{\sin\frac{jt}{p}e_{k,}\cos\frac{jt}{p}e_{k,}k=1,2\dots,N\right\},$$

where $e_1=(1,0,\dots,0),\dots,e_N=(0,\dots,0,1)\in\mathbf{R}^N$.

Now we define the dual functional (see [6])

$$I^*(u)=\int_0^{2\pi p}\left[\frac{1}{2}\mathscr{K}u\cdot u+F^*(t,u)\right]dt,\tag{2.4}$$

$\forall u\in X$. Its subgradient is

$$\partial_X I^*(u)=\{w\in X^*\mid-\mathscr{K}u(t)+\rho+w(t)\in\partial F^*(t,u(t)),\ \text{a.e.},\rho\in\mathbf{R}^N\},$$

where $\partial F^*(t,u)$ is the subgradient of $F^*(t,\cdot)$ at $u\in\mathbf{R}^N$ in the usual sense:

$$\partial F^*(t,u)=\{\xi\in\mathbf{R}^N\mid F^*(t,v)-F^*(t,u)\geq(\xi,v-u),\quad\forall v\in\mathbf{R}^N\}.$$

If $u^*\in X$ is a critical point of I^*, i.e., $\theta\in\partial_X I^*(u^*)$, then for some $\rho\in\mathbf{R}^N$, $-\mathscr{K}u^*(t)+\rho\in\partial F^*(t,u^*(t))$ a.e. From the reciprocity formula, we have

$$x\in\partial F^*(t,u)\iff F(t,x)+F^*(t,u)=(x,u)\iff u=F_x(t,x).$$

Thus, $x(t) = -\mathscr{K}u^*(t) + \rho$ is a $2\pi p$-periodic function and satisfies

$$\frac{d^2}{dt^2}x(t) = -u^*(t) = -F_x(t, x(t)) = -f(t, x(t), \ldots, x(t)). \tag{2.5}$$

Note that $x(t - r_j) = x(t - q(\tau_j/\tau)(2\pi p)) = x(t)(j = 1, 2, \ldots, s)$, as τ_j/τ is an integer. By (2.5), we obtain

$$\frac{d^2}{dt^2}x(t) + f(t, x(t - r_1) \ldots, x(t - r_s)) = 0.$$

Consequently, $x(t)$ is a $2\pi p$ periodic solution of (2.1). This reduces the problem to finding critical points (with minimal period $2\pi p$) for I^*.

In order to employ a min-max type argument to produce multiple critical points for the functional I^*, recall the Z_p version of the deformation lemma (see [5, 6]):

LEMMA 2.1. *Let $\Phi \in C^1(X, \mathbf{R})$ be a μ-invariant functional satisfying the* (P.S.) *condition. For any real number $c \in \mathbf{R}$ and any neighborhood U of $K_c = \{u \in X \mid \Phi(u) = c, \theta \in \partial_X \Phi(u)\}$, there exists a constant $\bar{\varepsilon} > 0$ such that for any $0 < \varepsilon < \bar{\varepsilon}$, there is a continuous map η: $X \to X$ such that*

(a) *η is a μ-equivariant homeomorphism*;

(b) *$\eta(u) \in \Phi_{c-\varepsilon}, \forall u \in \Phi_{c+\varepsilon} \backslash U$.*

If $K_c = \emptyset$, then $\eta(u) \in \Phi_{c-\varepsilon}, \forall u \in \Phi_{c+\varepsilon}$, where $\Phi_h = \{u \in X \mid \Phi(u) \leq h\}$.

A full statement and proof of the Z_p-deformation theorem can be found in [8] with obvious modification. By means of the approaches used in [5, 12], we have the following lemma:

LEMMA 2.2. *For integer j, define*

$$c_j = \inf_{\substack{i_m(A) \geq j \\ A \in \sum}} \sup_{u \in A} I^*(u), \tag{2.6}$$

where the meaning of m can be found in formulas (1.4) *and* (1.5). *Assume $-\infty < c_j < +\infty$; then c_j is a critical value of I^*. And if $c = c_j = c_{j+1} = \cdots = c_{j+l-1}$, then $i_m(K_c) \geq l$, where $K_c = \{u \in X \mid I^*(u) = c, \theta \in \partial_X I^*(u)\}$.*

We are now in a position to finish the proof of Theorem 2.1.

Proof of Theorem 2.1. According to Lemma 2.2, the following three claims remain to be proven:

Claim I. Under the assumption of Theroem 2.1, we have

$$-\infty < c_1 \leq \cdots \leq c_{2nN} < +\infty.$$

In fact, for any $u(t) \in X$, we may express $u(t)$ as

$$u(t) = \sum_{|j|=1}^{+\infty} \xi_j e^{i(j/p)t},$$

where $\xi_j \in \mathbf{C}^N$ and $\xi_{-j} = \bar{\xi}_j$.

If $u(t)$ is $\frac{2\pi p}{l}$-periodic, l a positive integer, then

$$u(t) = \sum_{|j|=1}^{+\infty} \xi_{lj} e^{i(lj/p)t}. \tag{2.7}$$

Therefore,

$$\begin{aligned} I^*(u(t)) &= \int_0^{2\pi p} \left[\frac{1}{2}\mathcal{K} u(t) \cdot u(t) + F^*(t, u(t))\right] dt \\ &= \frac{1}{2}\left(-\sum_{|j|=1}^{+\infty} \frac{p^2}{l^2 j^2} |\xi_{lj}|^2\right)(2\pi p) + \int_0^{2\pi p} F^*(t, u(t)) dt \\ &\geq -\frac{1}{2}\frac{p^2}{l^2}\|u\|_{L^2}^2 + \frac{1}{\alpha}\left(\frac{1}{\alpha_2}\right)^{\alpha-1} \|u\|_{L^\alpha}^\alpha \\ &\geq -\frac{1}{2}\frac{p^2}{l^2}(2\pi p)^{(\alpha-2/\alpha)}\|u\|_{L^\alpha}^2 + \frac{1}{\alpha}\left(\frac{1}{\alpha_2}\right)^{\alpha-1} \|u\|_{L^\alpha}^\alpha \\ &\geq -\left(\frac{1}{2} - \frac{1}{\alpha}\right)(2\pi p)\left(\frac{p^2}{l^2}\right)^{\alpha/(\alpha-2)} \left(\alpha_2^2\right)^{(\alpha-1)/(\alpha-2)}. \end{aligned}$$

So

$$I^*(u(t)) \geq -\left(\frac{1}{2} - \frac{1}{\alpha}\right)(2\pi p)\left(\frac{p^2}{l^2}\right)^{\alpha/(\alpha-2)} \left(\alpha_2^2\right)^{(\alpha-1)/(\alpha-2)} \tag{2.8}$$

as one finds by minimizing with respect to $\|u\|_{L^\alpha}$.

We now define

$$\mathcal{B}_l = -\left(\frac{1}{2} - \frac{1}{\alpha}\right)(2\pi p)\left(\frac{p^2}{l^2}\right)^{\alpha/(\alpha-2)} \left(\alpha_2^2\right)^{(\alpha-1)/(\alpha-2)}. \tag{2.9}$$

From the inequality (2.8) we obtain

$$\mathcal{B}_l \leq \inf_{u \in X} I^*(u) \leq c_j, \qquad j = 1, 2, \ldots, 2nN.$$

Now we only need to check $c_{2nN} < +\infty$. For any $\varepsilon > 0$, consider the invariant subset $S_{n, n_\varepsilon, \varepsilon}$ (given by (1.7)), with

$$\eta_{\min} = \left(\frac{\alpha_1}{2}\right)^{(\alpha-1)/(\alpha-2)} p^{2/(\alpha-2)}$$

and

$$\eta_\varepsilon = \eta_{\min} - \varepsilon > 0.$$

By Theorem 1.4, we have

$$i_m(S_{n,\, n_\varepsilon,\, \varepsilon}) = 2nN.$$

Let $u(t) = \sum_{|j|=1}^{n} \xi_j e^{i(jt)/p} \in S_{n,\, n_\varepsilon,\, \varepsilon}$ with $|\xi_1| = \eta_\varepsilon$ and $\sum_{j=2}^{n} |\xi_j| = \varepsilon$. We get

$$\begin{aligned}
I^*(u(t)) &= \int_0^{2\pi p} \frac{1}{2}\mathscr{K}u(t)\cdot u(t)dt + \int_0^{2\pi p} F^*(t, u(t))dt \\
&= \frac{1}{2}\left(-2\sum_{j=1}^{n}\frac{p^2}{j^2}|\xi_j|^2\right)(2\pi p) + \int_0^{2\pi p} F^*(t, u(t))dt \\
&\le \frac{1}{2}(-2p^2|\xi_1|^2)(2\pi p) + \frac{1}{\alpha}\left(\frac{1}{\alpha_1}\right)^{\alpha-1}\left(2\sum_{j=1}^{n}|\xi_j|\right)^{\alpha}(2\pi p) \\
&\le -p^2(\eta_{\min}-\varepsilon)^2(2\pi p) + \frac{1}{\alpha}\left(\frac{1}{\alpha_1}\right)^{\alpha-1}(2\eta_{\min})^{\alpha}(2\pi p) \\
&= -p^2\eta_{\min}^2(2\pi p) + \frac{1}{\alpha}\left(\frac{1}{\alpha_1}\right)^{\alpha-1}(2\eta_{\min})^{\alpha}(2\pi p) \\
&\quad - p^2(\varepsilon^2 - 2\eta_{\min}\varepsilon)(2\pi p) \\
&= -\left(\frac{1}{2}-\frac{1}{\alpha}\right)(2\pi p)(p^2)^{\alpha/(\alpha-2)}(\alpha_1^2)^{(\alpha-1)/(\alpha-2)}2^{-\alpha/(\alpha-2)} \\
&\quad - p^2(\varepsilon^2 - 2\eta_{\min}\varepsilon)(2\pi p).
\end{aligned}$$

Thus, recalling $i_m(S_{n,\, \eta_\varepsilon,\, \varepsilon}) = 2nN$, $\forall \varepsilon > 0$, we have

$$\begin{aligned}
c_{2nN} &= \inf_{\substack{i_m(A)\ge 2nN \\ A\in\Sigma}} \sup_{u\in A} I^*(u) \\
&\le -\left(\frac{1}{2}-\frac{1}{\alpha}\right)(2\pi p)(p^2)^{\alpha/(\alpha-2)}(\alpha_1^2)^{(\alpha-1)/(\alpha-2)}2^{-\alpha/(\alpha-2)} \\
&\quad - p^2(\varepsilon^2 - 2\eta_{\min}\varepsilon)(2\pi p).
\end{aligned}$$

Since ε is arbitrary, it follows that

$$c_{2nN} \le -\left(\frac{1}{2}-\frac{1}{\alpha}\right)(2\pi p)(p^2)^{\alpha/(\alpha-2)}(\alpha_1^2)^{(\alpha-1)/(\alpha-2)}2^{-\alpha/(\alpha-2)} < +\infty. \tag{2.10}$$

Claim II. If $u^* \in K_{c_j}$, $j = 1, 2, \ldots, 2nN$, then the minimal period of u^* is $2\pi p$.

In fact, if $u^* \in K_{c_j}$, then $I^*(u^*) = c_j$. From (2.10) and $\alpha_2/\alpha_1 < (S_p/\sqrt{2})^\beta$, we know that

$$\begin{aligned} I^*(u^*) &\le -\left(\frac{1}{2} - \frac{1}{\alpha}\right)(2\pi p)(p^2)^{\alpha/(\alpha-2)}(\alpha_1^2)^{(\alpha-1)/(\alpha-2)}2^{-\alpha/(\alpha-2)} \\ &< -\left(\frac{1}{2} - \frac{1}{\alpha}\right)(2\pi p)(p^2)^{\alpha/(\alpha-2)}\left(\alpha_2^2\left(\frac{2}{S_p^2}\right)^\beta\right)^{(\alpha-1)/(\alpha-2)}2^{-\alpha/(\alpha-2)} \\ &\le -\left(\frac{1}{2} - \frac{1}{\alpha}\right)(2\pi p)\left(\frac{p^2}{S_p^2}\right)^{\alpha/(\alpha-2)}(\alpha_2^2)^{(\alpha-1)/(\alpha-2)}. \end{aligned} \tag{2.11}$$

Assume that u^* has the minimal period $(2\pi p)/l$ for some integer $l \ge 1$. Comparing the inequality (2.8) with (2.11), we get $l < S_p$. Since u^* is a critical point for I^*, then for some $\rho \in \mathbf{R}^N$, $x(t) = \mathscr{K}u^*(t) + \rho \in \partial F^*(t, u^*(t))$ satisfies

$$\frac{d^2}{dt^2}x(t) + F_x(t, x(t)) = 0$$

and has minimal period $(2\pi p)/l$. But by condition (H_3), p/l is necessarily an integer. Since $l < S_p$, it must be that $l = 1$. Therefore, the minimal period of $u^*(t)$ is $2\pi p$.

Claim III. If $c = c_j = \cdots = c_{j+l-1}$, $1 \le j < j + l - 1 \le 2nN$, then there exist in K_c infinitely many distinct critical orbits.

In fact assume that

$$K_c = \sum_{k=1}^{s}[u_k^*(t)],$$

where $[u_k^*(t)]$ is the Z_p orbit of $u_k^*(t)$.

Since the minimal period of $u_k^*(t)$ is $2\pi p$,

$$\mu^{j_1}u_k^*(t) \ne \mu^{j_2}u_k^*(t), \qquad 0 \le j_1 < j_2 \le p - 1.$$

Hence we can define the map φ: $K_c \to Y\backslash\{\theta\}$ as follows:

$$\varphi(\mu^j u_k^*(t)) = e^{ij(2\pi)/p}, \qquad j = 0, 1, \ldots, p-1, \qquad k = 1, 2, \ldots, s.$$

It is easy to verify that

$$\varphi(\mu\mu^j u_k^*(t)) = e^{i(2\pi)/p}\varphi(\mu^j u_k^*(t)), \quad 0 \le j \le p-1, \quad k = 1, 2, \ldots, s.$$

Therefore, $i_m(K_c) = 1$. This is a contradiction to the results of Lemma 2.2.

The conclusion of Theorem 2.1 follows from these claims immediately. Indeed, from Lemma 2.2 and Claim I, we have $2nN$ critical values. If some of them are the same, Claim III gives infinitely many distinct critical points. And Claim II proves that all solutions we obtained have the minimal period of $2\pi p$. This concludes the proof of Theorem 2.1.

Without assumption (H_3), we have the following theorem

THEOREM 2.2. *Under hypotheses* (H_1) *and* (H_2) *with* $\alpha_2/\alpha_1 < 2^{\beta/2}$, *for any given positive integer* p, *if* $p \mid \tau$, *then* (2.1) *admits at least* $2nN$ *distinct solutions with minimal period* pT.

The theorem can be proved essentially by the same arguments. We need only consider the Claim II. In this case, (2.11) becomes

$$I^*(u^*) \leq -\left(\frac{1}{2} - \frac{1}{\alpha}\right)(2\pi p)\left(\frac{p^2}{2^2}\right)^{\alpha/(\alpha-2)}(\alpha_2^2)^{(\alpha-1)/(\alpha-2)} = \mathscr{B}_2$$

for all $u^* \in K_{c_j}$, $j = 1, 2, \ldots, 2nN$. Thus, by (2.8), all critical points we obtained have minimal period $2\pi p$.

3. FINAL REMARKS

We conclude the paper with some remarks.

Remark 1. Several papers (cf. [1, 4, 7] and references therein) have given some sufficient conditions for the existence of periodic solutions of autonomous functional differential equations. However, there are few papers dealing with nonautonomous functional differential systems. Although we have discussed in this paper a certain kind of second order functional differential equation, our method can be used in rather general cases with some obvious modifications. For example, we can investigate the system

$$\frac{dx(t)}{dt} = f(t, x(t - r_1), \ldots, x(t - r_s)), \tag{3.1}$$

where $x = (x_1, \ldots, x_{2N}) \in \mathbf{R}^{2N}$,

$$f\big(t + T, x^{(1)}, \ldots, x^{(s)}\big) = f\big(t, x^{(1)}, \ldots, x^{(s)}\big), \qquad \forall x^{(j)} \in \mathbf{R}^{2N},$$

and $r_j = \tau_j T$, τ_j are integers, $j = 1, 2, \ldots, s$.

For this instance, we shall give discussion in another paper.

Remark 2. In [5, 6], the authors obtain a multiplicity result for a certain class of second order time periodic systems, namely, for the problem

$$\frac{d^2}{dt^2}x(t) + f(t, x) = 0, \qquad x(0) = x(pT), \tag{3.2}$$

where $x = (x_1, x_2, \ldots, x_N) \in \mathbf{R}^N$, $p > 1$ is an integer, $f(t + T, x) = f(t, x)$, $\forall x \in \mathbf{R}^N$, $\forall t \in \mathbf{R}$.

They established the following result: Let f satisfy the assumptions similar to (H_2), (H_3); then for any integer $p > 1$ satisfying

$$\frac{\alpha_2}{\alpha_1} < \left(\frac{3S_p^2}{n(n+1)(2n+1)}\right)^{\beta/2} \tag{3.3}$$

for some integer $n \geq 1$, problem (3.2) admits, at least, $2nN$ (in [5]) or nN (in [6]) distinct solutions with minimal period pT. From this result, the number of periodic solutions of (3.2) with minimal period pT tends towards infinity as S_p tends towards infinity. However, if S_p remains bounded as $p \to \infty$ (e.g., $p = 2^k$ and $S_p = 2$), one does not even know whether (3.2) admits solutions with minimal period pT.

Now, from our Theorem 2.1, we have

THEOREM 3.1. *Let f satisfy* (H_2), (H_3). *Then for any given integer $p > 1$, satisfying*

$$\frac{\alpha_2}{\alpha_1} < \left(\frac{S_p^2}{2}\right)^{\beta/2},$$

problem (3.2) *admits at least $2nN$ distinct solutions with minimal period pT, where*

$$n = \max\{l \in \mathbf{N} \mid [\mathrm{scm}(l)]^{lN} \neq 0 (\mathrm{mod}\, p)\}.$$

Note that when S_p remains bounded as $p \to \infty$, it is obvious that there exists a sequence of positive integers $\{n_p\}$ such that $[\mathrm{scm}(n_p)]^{n_p N} \neq 0(\mathrm{mod}\, p)$ and $n_p \to \infty$ as $p \to \infty$. On the other hand, according to the result of [5, 6], if (3.3) is true for n, then $\frac{n(n+1)(2n+1)}{3} < S_p^2$; thus $n \leq S_p - 1$. Therefore, our result has improved the results of [5, 6] and it gives a much better conclusion.

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